THE QUASI-STATE SPACE OF A UNITAL C^*-ALGEBRA IS A TOPOLOGICAL QUOTIENT OF THE REPRESENTATION SPACE

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Abstract

We show that for any unital C^*-algebra A, a sufficiently large Hilbert space H and a unit vector $\xi \in H$, the natural application $rep(A:H) \xrightarrow{\theta_\xi} Q(A)$, $\pi \mapsto \langle \pi(-)\xi, \xi\rangle$ is a topological quotient, where $rep(A:H)$ is the space of possibly degenerate representations on H and $Q(A)$ the set of quasi-states, i.e. positive linear functionals with norm at most 1. This quotient allows to prove as a simple corollary Takesaki-Bichteler duality for arbitrary C^*-algebras. We clarify some aspects about the relevant concept of field over the representations of a C^*-algebra A. Finally, we give a description of the multipliers of A in terms of fields.

1 Introduction

The elements of a C^*-algebra A can be thought as *fields* over the space of representations on a Hilbert space H. A field is a bounded map $rep(A:H) \to B(H)$ satisfying a compatibility condition. An element $a \in A$ defines $\widehat{a}(\pi) := \pi(a)$. M. Takesaki [5] proved in 1967 that, for large enough H, the set of these fields form the universal von Neumann algebra of A and, in the separable case, that the elements of A can be characterized as those fields continuous with respect to the right topologies. This is Takesaki duality for C^*-algebras. Two years later, K. Bichteler [1] generalized the duality for arbitrary C^*-algebras[1]. Here we provide a stronger and simpler

[1]See [4] for a brief survey on duality theorems for C^*-algebras and a duality theorem where the dual object is the irreducible representation space.

statement: (theorem 2.4) for unital A, the application $rep(A : H) \xrightarrow{\theta_\xi} Q(A)$, $\theta_\xi(\pi) = \langle \pi(-)\xi, \xi \rangle$ is a topological quotient, where $\xi \in H$ is a unit vector. Takesaki-Bichteler duality follows immediately from this quotient (section 3, 3.11). The statement of our theorem 2.4 is simpler than Takesaki-Bichteler duality theorem because it predicates on more primitive objects. However, the proof is strictly harder since it involves essentially the argument by Bichteler and further work.

In section 3 we give a detailed study of the concept of field. Subjectively, our definition 3.1 improves the definitions from [5] and [1]. Next we unfix the Hilbert space and consider the category $rep(A)$ of nondegenerate representations of A. In pedantic but concise terms, a field is a bounded endomorphism of the forgetful functor $rep(A) \to \mathcal{H}$, where $\mathcal{H}$ is the category of Hilbert spaces. This point of view is usual in the context of group representation theory. Here it was useful to understand better why it suffices with cyclic representations to define fields (proposition 3.5). Finally, we give a description of the different multiplier spaces $LM(A)$, $RM(A)$, $M(A)$, $QM(A)$ in terms of fields.

1.1 Notation

• A will denote a C^*-algebra.
• If X is a Banach space, X^* denote its dual.
• $S(A) = \{\varphi \in A^* / \varphi \geq 0,\ ||\varphi|| = 1\}$ the state space of A, with the weak-* topology.
• $Q(A) = \{\varphi \in A^* / \varphi \geq 0,\ ||\varphi|| \leq 1\}$, the space of quasi-states, also with the weak-* topology.
• For $\varphi \in A^*$, $\varphi \geq 0$, $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is the GNS triple. $||\xi_\varphi||^2 = ||\varphi||$.

2 Main theorem

Let $rep(A : H)$ be the set of possibly degenerate representations of A on H, this is the set of $*$-algebra morphisms $A \to B(H)$. Here H is a Hilbert space of an infinite dimension greater or equal than the dimension of every cyclic representation of A (these dimensions are bounded: for a representation π with cyclic vector ξ consider the dense subspace $\{\pi(a)\xi / a \in A\}$ and the fact that any dense subspace of a Hilbert space contains an orthonormal basis).

The relevant topology on $rep(A : H)$ is the pointwise convergence topology with respect to the *wot*, *sot*, σ-weak or σ-strong topologies in $B(H)$. Next lemma asserts that these topologies coincide.

Lemma 2.1. *Let π be a representation of A on a Hilbert space H and (π_j) a net of such representations. Convergence $\pi_j(a) \to \pi(a)$ for all $a \in A$ is equivalent for the wot, sot, σ-weak and σ-strong topologies on $B(H)$.*

See [1] for the proof (page 90). In other words, the topology we consider on $rep(A : H)$ is that inherited from the product topology on $B(H)^A$, where the topology on $B(H)$ can equivalently be the σ-weak, σ-strong, *wot* or *sot*. It is Hausdorff because it is a subspace of a product of Hausdorff spaces.

For the proof of theorem 2.4 we need two geometrical lemmas. The first of them says that if two n-tuples of vectors in a Hilbert space have similar orthogonality relations (inner products between the vectors of each tuple) then, up to an isometry, the n-tuples are close in norm. For the proof see lemma 3.5.6 of Dixmier's book [2].

Lemma 2.2. *Let $v_1, ..., v_n \in H$, H a Hilbert space. For every $\epsilon > 0$ there is a $\delta > 0$ such that for every $w_1, ..., w_n \in H$ with $|\langle w_i, w_j\rangle - \langle v_i, v_j\rangle| < \delta$ $\forall i, j$ there is a unitary operator $H \xrightarrow{U} H$ such that $||U(w_i) - v_i|| < \epsilon$ $\forall i$.*

Lemma 2.3. *Let H be a Hilbert space and $\alpha, \beta \in H$ unit vectors. Then there is a unitary $U_{\alpha\to\beta}$ such that $||U_{\alpha\to\beta} - Id|| = ||\alpha - \beta||$.*

Proof. In case $\beta = k\alpha$ for $k \in \mathbb{C}$, then $|k| = 1$ and $U_{\alpha\to\beta} := k.Id$. Otherwise, we define $U_{\alpha\to\beta}$ as the identity on $[\alpha]^\perp \cap [\beta]^\perp = [\alpha, \beta]^\perp$. On the subspace $[\alpha, \beta]$ we take an orthonormal basis (α, α'). Write $\beta = r\alpha + s\alpha'$, and $\beta' := -\overline{s}\alpha + \overline{r}\alpha'$, obtaining an orthonormal basis (β, β'). Now define $U_{\alpha\to\beta}|_{[\alpha,\beta]}$ by $\alpha \mapsto \beta$, $\alpha' \mapsto \beta'$. For $x \in H$, let $\lambda\alpha + \mu\alpha'$ be the projection of x to $[\alpha, \beta]$. We have:

$$||x - U_{\alpha\to\beta}(x)||^2 = ||\lambda\alpha + \mu\alpha' - \lambda\beta - \mu\beta'||^2 =$$
$$= ||\lambda\alpha + \mu\alpha' - \lambda(r\alpha + s\alpha') - \mu(-\overline{s}\alpha + \overline{r}\alpha')||^2 = ... = (|\lambda|^2 + |\mu|^2)||\alpha - \beta||^2$$

So $||x - U_{\alpha\to\beta}(x)|| = ||\alpha - \beta||.||p_{[\alpha,\beta]}(x)|| \le ||\alpha - \beta||.||x||$. □

Theorem 2.4. *Let A be a unital C^*-algebra and H a Hilbert space of dimension d, greater or equal than the dimension of any cyclic representation of A, plus 1[2]. Let $\xi \in H$ be a unit vector. Then,*

[2] Of course, the "+1" only has effect in the easy case $d < \infty$.

a) the application

$$rep(A:H) \xrightarrow{\theta_\xi} Q(A)$$
$$\pi \longmapsto \langle \pi(-)\xi, \xi \rangle$$

is a quotient map.

b) The restriction $rep_\xi(A:H) \xrightarrow{\theta_\xi} S(A)$ *is a quotient, where* $rep_\xi(A:H) = \{\pi \in rep(A:H) / \xi \in \pi(1)H\}$. *(Recall* $\pi(1)H = \overline{\pi(A)H}$*).*

Proof.

a) Continuity is trivial. Each $\varphi \in Q(A)$ has many preimages. To produce a preimage of φ_0 we must embed a GNS representation of φ_0 in H in such a way that the orthogonal projection of ξ to the essential space is the cyclic vector of the GNS. To achieve this, take a unit vector η orthogonal to ξ, define $\xi_0 = ||\varphi_0||\xi + (||\varphi_0|| - ||\varphi_0||^2)^{\frac{1}{2}}\eta$. This ξ_0 satisfies $||\xi_0||^2 = ||\varphi_0||$ and $\xi - \xi_0 \perp \xi_0$. By hypothesis, it is possible to embed H_{φ_0} into $[\xi - \xi_0]^\perp$ taking ξ_{φ_0} to ξ_0 (notice that, in the case d infinite, it is possible to embed H_{φ_0} in a way such that its codimension is also d). Define $\pi_0 \in rep(A:H)$ as π_{φ_0} through the isometry $H_{\varphi_0} \hookrightarrow H$, being 0 on the orthogonal to the image of H_{φ_0}. We have $\theta_\xi(\pi_0) = \varphi_0$.

In case d is finite, $rep(A:H)$ is compact, so θ_ξ is closed and therefore a quotient. To see compactness of $rep(A:H)$, consider the map $rep(A:H) \to B_1^{A_1}$, $\pi \mapsto (a \mapsto \pi(a))$, where $B_1 \subset B(H)$ and $A_1 \subset A$ are the respective unit balls. $B_1^{A_1}$ has the product topology of the norm topology in B_1, it is a compact space. The map is a topological subspace and the image is closed.[3]

For infinite d, we divide in four parts the proof that $Q(A)$ has the quotient topology.

<u>Part 1:</u>

Take $D \subset Q(A)$ such that $\theta_\xi^{-1}(D)$ is open. We must see that D is open to conlude that θ_ξ is a quotient. Let $\varphi_0 \in D$. Take a preimage π_0 of φ_0 as before, such that $\pi_0(1)H = \overline{\pi_0(A)\xi_0}$ has codimension d. Now take a basic open neighborhood V of π_0 contained in $\theta_\xi^{-1}(D)$:

$$V = \{\pi \in rep(A:H) / ||\pi(c_j)\alpha_i - \pi_0(c_j)\alpha_i|| < \epsilon \ \forall i = 1, ..., m;\ j = 1, ..., n\}$$

[3] In the infinite dimensional case, taking the *wot* on B_1, the image apparently fails to be closed.

We shall find an open neighborhood of φ_0 (W below) such that every element φ in that neighborhood is the image of an element $\pi \in V$. This would finish the proof.

Part 2: We decompose $\alpha_i = \beta_i + \gamma_i$ where $\beta_i \in \pi_0(1)H$, $\gamma_i \in (\pi_0(1)H)^\perp$. To obtain $\pi \in V$ we will satisfy $||\pi(c_j)\beta_i - \pi_0(c_j)\beta_i||$ arbitrarily small, and the same for γ_i.

To obtain $||\pi(c_j)\beta_i - \pi_0(c_j)\beta_i||$ arbitrarily small, we can approximate β_i with $\pi_0(b_i)\xi_0$, so it will suffice with $||\pi(c_j)\pi_0(b_i)\xi_0 - \pi_0(c_j)\pi_0(b_i)\xi_0||$ arbitrarily small.

Now take the following open neighborhood of φ_0

$$W = \{\varphi \in Q(A) / |(\varphi - \varphi_0)(b_k^* c_l^* c_j b_i)| < \delta \ \forall i = 0, ..., m; j = 0, ..., n\}$$

where $b_0 = c_0 = 1$. Thus, for $\varphi \in W$, the orthogonality relations of the set $\{\pi_\varphi(c_j)\pi_\varphi(b_i)\xi_\varphi\}_{\substack{i=0,\dots,m\\ j=0,\dots,n}}$ are similar to those of $\{\pi_0(c_j)\pi_0(b_i)\xi_0\}_{\substack{i=0,\dots,m\\ j=0,\dots,n}}$. Hence, there is an isometric embedding $H_\varphi \hookrightarrow H$ such that the images of $\pi_\varphi(c_j b_i)\xi_\varphi$ are close in norm to $\pi_0(c_j b_i)\xi_0$ (first choose any isometric embedding and then use lemma 2.2). Let us call $\pi' \in rep(A : H)$ the representation π_φ through the embedding, being 0 on the orthogonal complement of the image of H_φ; let ξ' be the image of ξ_φ. We have:

$$\begin{gathered} ||\pi'(c_j)\pi_0(b_i)\xi_0 - \pi_0(c_j)\pi_0(b_i)\xi_0|| \leq \\ \leq ||\pi'(c_j)\pi_0(b_i)\xi_0 - \pi'(c_j)\pi'(b_i)\xi'|| + ||\pi'(c_j b_i)\xi' - \pi_0(c_j b_i)\xi_0|| \leq \\ \leq \max_j ||c_j||.||\pi_0(b_i)\xi_0 - \pi'(b_i)\xi'|| + ||\pi'(c_j b_i)\xi' - \pi_0(c_j b_i)\xi_0|| \leq \epsilon \end{gathered}$$

So far we achieved "$||\pi(c_j)\beta_i - \pi_0(c_j)\beta_i||$ arbitrarily small". Notice that π' (on its essential space) is a GNS for φ but it is not necessarily true that $\theta_\xi(\pi') = \varphi$. We will correct this in part 3.

To obtain "$||\pi(c_j)\gamma_i - \pi_0(c_j)\gamma_i||$ arbitrarily small", we can simply embed $H_\varphi \hookrightarrow H' \subset H$ in the previous procedure, where $H' = H \cap [\gamma_i]^\perp$, so the resulting π' has essential space orthogonal to every γ_i, and $\pi'(c_j)\gamma_i - \pi_0(c_j)\gamma_i = 0$ ($j = 1, ..., n$). Actually, for the next part, we will also need $\xi - \xi_0 \perp \pi'(1)H$, so instead of H' we embed H_φ inside $H'' = H' \cap [\xi - \xi_0]^\perp$ for previous procedure (this is possible because $\pi_0(1)H \subset H''$).

Part 3:

We will rotate π' slightly to a representation π so that $\theta_\xi(\pi) = \varphi$. First we need a unit vector η close to ξ such that $\eta - \xi' \perp \pi'(1)H$.

In case $\xi = \xi_0$, since ξ' is close to ξ_0 and $||\xi'|| \leq ||\xi|| = 1$, we can take a small $v \in H$ orthogonal to $\pi'(A)H$ such that $||\xi' + v|| = 1$, and define $\eta := \xi' + v$.

In case $\xi \neq \xi_0$, we take $\eta = \lambda(\xi - \xi_0) + \xi'$. To determine λ:

$$||\eta||^2 = |\lambda|^2.||\xi - \xi_0||^2 + ||\xi'||^2 = |\lambda|^2.(1 - ||\xi_0||^2) + ||\varphi|| = |\lambda|^2.(1 - ||\varphi_0||) + ||\varphi||$$

so we choose $\lambda = \frac{1-||\varphi||}{1-||\varphi_0||}$ to obtain $||\eta|| = 1$. Since $\varphi(1)$ is close to $\varphi_0(1)$, λ is close to 1 and therefore η is close to ξ (i.e.: η is arbitrarily close to ξ as long as δ is sufficiently small).

Now, having η we just apply $U := U_{\eta \to \xi} \in U(H)$ (lemma 2.3), and take $\pi(-) := U^{-1}\pi'(-)U$. Since $||U - Id|| = ||\eta - \xi||$, we still have "$||\pi(c_j)\alpha_i - \pi_0(c_j)\alpha_i||$ arbitrarily small", so $\pi \in V$ and $\theta_\xi(\pi) = \varphi$.

b) Clearly we have the restriction $rep_\xi(A : H) \xrightarrow{\theta_\xi} S(A)$. Furthermore $\theta_\xi^{-1}(S(A)) = rep_\xi(A : H)$. Let $D \subset S(A)$ be a set such that $\theta_\xi^{-1}(D)$ is open in $rep_\xi(A : H)$, so $\theta_\xi^{-1}(D) = U \cap rep_\xi(A : H)$ with U open in $rep(A : H)$. Let $\varphi_0 \in D$. We take a preimage π_0 as before. We already know that $\theta_\xi(U)$ contains an open neighborhood $W \ni \varphi_0$, W open in $Q(A)$. Now it is easy to check:

$$W \cap S(A) \subset \theta_\xi(U \cap rep_\xi(A : H)) \subset D$$

Thus, D is open in $S(A)$. $\square$

We leave open the question of whether $rep(A : H) \xrightarrow{\theta_\xi} Q(A)$ is a quotient for nonunital C^*-algebras.

3 Fields

Here we review the concept of field used in Takesaki-Bichteler duality. We provide a more elegant definition than those from [5] and [1], and also consider a definition without fixing a Hilbert space. We explain why all these definitions are equivalent, thus clarifying some aspects.

Definition 3.1. *A field over $rep(A : H)$ is a map $rep(A : H) \xrightarrow{T} B(H)$ that satisfies:*
0) $T(0) = 0$
1) $\{||T(\pi)||\}_{\pi \in rep(A:H)}$ is bounded.
2) For an intertwiner $H \xrightarrow{S} H$ between π_1 and π_2 ($S\pi_1(a) = \pi_2(a)S \ \forall a \in A$), it holds $ST(\pi_1) = T(\pi_2)S$. In other words: T is compatible with intertwiners.

Clearly, every $a \in A$ induces a field.

Proposition 3.2. *The following condition is equivalent to item 2 in previous definition.*
2') T is compatible with intertwiners that are partial isometries.

Proof. 2) $\Rightarrow$ 2$'$) is trivial, let's prove the converse. Assume that T is compatible with intertwiners that are partial isometries and take an arbitrary intertwiner $\pi_1 \xrightarrow{S} \pi_2$, $S \in B(H)$. The operator S has a polar decomposition $S = UP$, where $P = (S^*S)^{\frac{1}{2}}$ and U maps $(S^*S)^{\frac{1}{2}}y$ to Sy and the orthogonal complement to 0. Since S is an intertwiner, $\pi_2 \xrightarrow{S^*} \pi_1$ is an intertwiner and also are $\pi_1 \xrightarrow{P} \pi_1$ and $\pi_1 \xrightarrow{U} \pi_2$. T is compatible with U by hypothesis. It only remains to prove that T is compatible with any positive intertwiner P of a representation π_1 with itself. Taking $r > 0$ small enough, rP has its spectrum inside $[0, 2\pi)$. e^{irP} is a unitary equivalence, so it is compatible with T (i.e. it commutes with $T(\pi_1)$). But rP is the logarithm of e^{irP}, so rP also commutes with $T(\pi_1)$.

□

Proposition 3.3. *Let $\pi \in rep(A : H)$, p_π the orthogonal projection to the essential space of π and T a field over $rep(A : H)$. Then $T(\pi) = p_\pi T(\pi) p_\pi$.*

Proof. Let $p_{\pi^\perp} = 1 - p_\pi$, the orthogonal projection to $\overline{\pi(A)H}^\perp$. It defines an intertwiner $\pi \xrightarrow{p_{\pi^\perp}} 0$, so we have $(1-p_\pi)T(\pi) = T(0)p_{\pi^\perp} = 0$, $T(\pi) = p_\pi T(\pi)$. Besides, p_π is an endomorphism of π, so $p_\pi T(\pi) = T(\pi)p_\pi$. □

3.1 Fields over representations as a category

Let $rep(A)$ be the category of nondegenerate representations of A with bounded intertwiners as arrows, and $cyc(A)$ the full subcategory of cyclic representations. The zero representation on $\{0\}$ is nondegenerate and cyclic.

Definition 3.4. *For a C^*-algebra A, we call "field over $rep(A)$" a function T assigning to each $\pi \in rep(A)$, $A \xrightarrow{\pi} B(H_\pi)$, an element $T(\pi) \in B(H_\pi)$ in a bounded and coherent with morphisms way. Explicitly: $\sup_\pi ||T(\pi)|| < \infty$, and if $H_{\pi_1} \xrightarrow{S} H_{\pi_2}$ is an intertwiner ($S\pi_1(a) = \pi_2(a)S$) then $ST(\pi_1) = T(\pi_2)S$. In other words, "fields over $rep(A)$" are bounded endomorphisms of the forgetful functor $rep(A) \to \mathcal{H}$, where $\mathcal{H}$ is the category of Hilbert spaces.*

Proposition 3.5. *The set of fields over $rep(A)$ is equal to the set of fields over $cyc(A)$.*

Of course, the definition of field over $cyc(A)$ is analogous to 3.4.

Proof. Clearly, a field over $rep(A)$ can be restricted to a field over $cyc(A)$. Now let T be a field over $cyc(A)$, and $(\pi, H) \in rep(A)$. π can be expressed as a direct sum of cyclic representations, so we define $T(\pi)$ as the direct sum of the operators associated to these subrepresentations. This definition is correct because of the following. Assume we have two decompositions into cyclic subrepresentations: $H = \bigoplus A_i = \bigoplus B_j$. Consider P_i and Q_j the orthogonal projections to the subspaces A_i and B_j. We have the following morphisms of cyclic representations, $B_j \xrightarrow{P_i|_{B_j}} A_i$. Compatibility of T says $T(A_i)P_i|_{B_j} = P_i|_{B_j}T(B_j)$ (we abuse harmlessly identifying the subspace with the subrepresentation).

$$\sum_i T(A_i)P_i = (\sum_i T(A_i)P_i)(\sum_j Q_j) = \sum_{i,j} T(A_i)P_iQ_j =$$

$$= \sum_{i,j} P_iT(B_j)Q_j = (\sum_i P_i)(\sum_j T(B_j)Q_j) = \sum_j T(B_j)Q_j$$

The sums converge strongly (*sot* in $B(H)$). It is valid to interchange the order of summation because composition of operators is jointly continuous for the strong operator topology when restricted to bounded sets. This proves that T is well defined.

The extended field is clearly bounded. To see compatibility, take a morphism between π_1 and π_2, $H_1 \xrightarrow{S} H_2$, and any vector $\alpha \in H_1$. Now take decompositions of these representations as sum of cyclic subrepresentations, containing the cyclic representations generated by α and $S(\alpha)$ respectively. S restricts to an intertwiner between these cyclic representations. Because of the original compatibility in $cyc(A)$, we have $ST(\pi_1)(\alpha) = T(\pi_2)S(\alpha)$. $\square$

Proposition 3.6. *There is a natural bijection between fields over $rep(A)$ and fields over $rep(A : H)$ for H a Hilbert space with dimension greater or equal than the dimension of every cyclic representation of A.*

Proof. Let $rep_0(A)$ be the category of possibly degenerate representations of A. Observe that a field over $rep(A)$ is equivalent to a field over $rep_0(A)$ that vanishes on the zero representation on nontrivial Hilbert spaces. Let T be a

field over $rep_0(A)$ vanishing on the zero representation. We define the field T' over $rep(A:H)$ as $T'(\pi) = T(\pi)$. It clearly satisfies conditions (0), (1) and (2). Now take T' a field over $rep(A:H)$. We will define a field T over $cyc(A)$. Let $\pi \in cyc(A)$ and $H_\pi \overset{i}{\hookrightarrow} H$ an isometry. We have a representation $\pi' \in rep(A:H)$ that acts like π on $i(H_\pi)$ and acts by 0 on the complement. By proposition 3.3 the range of $T'(\pi')$ is contained in $i(H_\pi)$, so we can define $T(\pi)$ as $T'(\pi')$ through the isometry i. For an intertwiner between cyclic representations $H_{\pi_1} \xrightarrow{S} H_{\pi_2}$ we embed both Hilbert spaces on H through i_1 and i_2 and define $H \xrightarrow{S'} H$ as $i_2 S i_1^{-1}$ S on $i_1(H_{\pi_1})$ and 0 on the complement. Compatibility of T with S follows from compatibility of T' with S'. $\square$

With the operations defined pointwise and the norm $||T|| = \sup_\pi ||T(\pi)||$, the set of fields is a C^*-algebra. Actually, they form the universal von Neumann algebra of A (see [5] theorem 3, [1] proposition in page 95, [6] proposition 4.7). Recall that the universal von Neumann algebra of a C^*-algebra A can also be constructed as the bicommutant of the universal representation $\bigoplus_{\varphi\in S(A)} \pi_\varphi$ or as the bidual A^{**} with the natural involution and Arens multiplication.

The definition of "field" by Takesaki can be summarized as follows: it is a bounded map $rep(A:H) \to B(H)$ with the property $T(\pi) = p_\pi T(\pi) p_\pi$ (see proposition 3.3) that preserves unitary equivalences and finite direct sums (for direct sums it is necessary to consider a unitary $H \oplus H \leftrightarrow H$). Our defintion is stronger because we have compatibility with arbitrary intertwiners and 3.3. The converse can be done through proposition 3.2: a field compatible with direct sums and unitary equivalences will be compatible with intertwiners that are partial isometries. We prefer not to write down the details. Actually, it is technically unnecessary, since we already know that all definitions lead to the enveloping von Neumann algebra of A.

Remark 3.7. The enveloping von Neumann algebra is a functor $\mathcal{C}^* \xrightarrow{W^*} \mathcal{W}^*$, where $\mathcal{C}^*$ denotes the category of C^*-algebras with $*$-algebra morphisms and $\mathcal{W}^*$ the category of W^*-algebras with normal morphisms. The functor W^* is left adjoint to the forgetful functor $\mathcal{W}^* \to \mathcal{C}^*$. This is the universal property of W^*. If we replace $\mathcal{C}^*$ by $\mathcal{G}r$, the category of topological groups with continuous group homomorphisms, we have a functor $\mathcal{G}r \xrightarrow{W^*} \mathcal{W}_1^*$ that is left adjoint to the unitary group functor $W_1^* \xrightarrow{U} \mathcal{G}r$ (here $\mathcal{W}_1^*$ is the category of W^*-algebras with unital normal morphisms). It is the universal W^*-algebra

of topological groups. In particular this extends Ernest's "big group algebra" construction [3] to arbitrary topological groups. See [6] (corollary 6.13) for more details.

3.2 Takesaki-Bichteler duality

This duality asserts that a C^*-algebra can be recovered as the set of continuous fields $rep(A:H) \to B(H)$, where the topology on $B(H)$ might be the σ-weak, σ-strong, *wot* or *sot*. Elements in A clearly induce continuous fields for all these topologies on $B(H)$. Since *wot* is the weakest among these, we have that *sot*-continuous, σ-weak-continuous and σ-strong-continuous fields are *wot*-continuous. Hence, it will suffice to prove that *wot*-continuous fields are elements of A. We denote the set of *wot*-continuous fields by $\mathcal{C}_0(rep(A:H))$. The subindex 0 emphasize that they annihilate on the zero representation.

In order to deduce the duality theorem from theorem 2.4, we need the following lemma taken from Bichteler's article ([1], first lemma, parts (iii) and (iv)).

Recall that any Banach space V can be recovered from the bidual as those elements $V^* \to \mathbb{C}$ that are continuous for the w^*-topology. This lemma in particular says that for a C^*-algebra A it suffices with continuity on $Q(A)$ instead of all A^*.

Definition 3.8. *Let $AN_0(Q(A))$ be the set of affine bounded $\mathbb{C}$-valued functions on $Q(A)$ taking the value 0 at 0. It is a normed space for the supremum norm. $AC_0(Q(A))$ will be the subspace of $AN_0(Q(A))$ of continuous functions.*

Lemma 3.9. *There is a Banach space isomorphism $A^{**} \to AN_0(Q(A))$ that restricts to a bijection $A \to AC_0(Q(A))$.*

For the proof see [1] first lemma or [6] (lemma 5.2) for a more thorough proof (also simpler for the second part).

Remark 3.10. Taking $S(A)$ instead of $Q(A)$ we have: $A^{**} \simeq AN(S(A))$ and, for unital A, $A \simeq AC(S(A))$ (where $AN(S(A))$ is the space of affine bounded $\mathbb{C}$-valued functions on $S(A)$ and $AC(S(A))$ the subspace of continuous functions).

Proof. It is straightforward to check that $AN_0(Q(A)) = AN(S(A))$. To obtain $AC_0(Q(A)) = AC(S(A))$ for unital A, we must prove that continuity on $S(A)$ implies continuity on $Q(A)$. So take $f \in AN_0(Q(A))$ continuous on $S(A)$ and $\varphi_\mu \to \varphi$ in $Q(A)$. Evaluating at 1, we have $||\varphi_\mu|| \to ||\varphi||$. If $\varphi = 0$ we have $|f(\varphi_\mu)| = ||\varphi_\mu||.|f(\frac{\varphi_\mu}{||\varphi_\mu||})| \leq ||\varphi_\mu||.||f||_\infty \to 0$ for those μ such that $\varphi_\mu \neq 0$ and $f(\varphi_\mu) = 0$ if $\varphi_\mu = 0$; so $f(\varphi_\mu) \to 0$. If $\varphi \neq 0$, for large enough μ we have $\varphi_\mu \neq 0$ and

$$f(\varphi_\mu) = ||\varphi_\mu||f(\frac{\varphi_\mu}{||\varphi_\mu||}) \to ||\varphi||f(\frac{\varphi}{||\varphi||}) = f(\varphi)$$

□

Theorem 3.11 (Takesaki-Bichteler duality)**.** *Every C^*-algebra A is isomorphic to the set of continuous fields $\mathcal{C}_0(rep(A : H))$, where H is a Hilbert space whose dimension is greater or equal to the dimension of any cyclic representation of A.*

Proof. We already know that an element of A defines a continuous field. Now take a *wot*-continuous field T, and assume $1 \in A$. Since fields form the universal W^*-algebra of A, by lemma 3.9 we have an element $f_T \in AN_0(Q(A))$ corresponding to T. According to the bijection between the set of fields and the bidual, we have $\langle T(\pi)\xi, \xi\rangle = f_T(\langle\pi(-)\xi, \xi\rangle)$, $\forall \pi \in rep(A : H)$, $\xi \in H$ unit vector. In other words, the following diagram commutes:

$$\begin{tikzcd}
rep(A : H) \arrow[r, "\theta_\xi"] \arrow[d, "T"'] & Q(A) \arrow[d, dashed, "f_T"] \\
B(H) \arrow[r, "\langle(-)\xi,\xi\rangle"] & \mathbb{C}
\end{tikzcd}$$

Since θ_ξ is a quotient (theorem 2.4), f_T is continuous, so, by lemma 3.9 it is an element of A.

As it was done historically in [5] and [1], we deduce the nonunital case from the unital case. Assume $1 \notin A$ and consider the minimal unitization $A \overset{i}{\hookrightarrow} \widetilde{A}$. Let T be a *wot*-continuous field on $rep(A : H)$. T defines a *wot*-continuous field $\widetilde{T}$ over $rep(\widetilde{A} : H)$, simply restricting representations of $\widetilde{A}$ to A. So $\widetilde{T} = (a, \lambda) \in \widetilde{A}$. But $\widetilde{T}(\rho) = 0$, where ρ is the trivial representation obtained by $\widetilde{A} \to \widetilde{A}/A \simeq \mathbb{C} \xrightarrow{(-)Id} B(H)$.

$$0 = \widetilde{T}(\rho) = \rho\big((a, \lambda)\big) = \lambda.Id$$

Thus $\lambda = 0$ and $T \in A$. □

Remark 3.12. For unital A, a field over $rep(A : H)$ only needs to be continuous on $rep_\xi(A : H)$ to be an element of A. This is because of part (b) of theorem 2.4 and remark 3.10.

3.3 Multipliers in terms of fields

In this context, the identity $M(C_0(X)) = C_b(X)$ for commutative C^*-algebras $C_0(X)$ motivates the idea of trying to express the multiplier algebra of A as certain continuous fields. The point at infinity corresponds to the zero representation. So the multipliers should be fields that are continuous except maybe at the 0 representation. We would like to use a definition of continuity for fields over $rep(A)$ (directly avoiding nondegenerate representations) to obtain $M(A) =$ "$\mathcal{C}_b(rep(A))$", but it is not clear whether this is possible. We will use instead the following notion: a field T will be "nondegenerately continuous" if for every convergent net of representations $\pi_j \to \pi$, $T(\pi_j)$ converges to $T(\pi)$ on the nondegenerate part of π, $\overline{\pi(A)H}$. More precisely:

Definition 3.13. *A field T over $rep(A : H)$ is called:*
- *s-nondegenerately continuous if for every convergent net $\pi_j \to \pi$ in $rep(A : H)$ and $\alpha \in \overline{\pi(A)H}$ we have $T(\pi_j)\alpha \to T(\pi)\alpha$.*
- *as-nondegenerately continuous if for every convergent net $\pi_j \to \pi$ in $rep(A : H)$ and $\alpha \in \overline{\pi(A)H}$ we have $T(\pi_j)^*\alpha \to T(\pi)^*\alpha$ ("as" stands for "antistrong").*
- *s^*-nondegenerately continuous if it is both s and as-nondegenerately continuous.*
- *w-nondegenerately continuous if for every convergent net $\pi_j \to \pi$ in $rep(A : H)$ and $\alpha, \beta \in \overline{\pi(A)H}$ we have $\langle T(\pi_j)\alpha, \beta\rangle \to \langle T(\pi)\alpha, \beta\rangle$.*

We denote with $\mathcal{C}^s(rep(A : H))$, $\mathcal{C}^{as}(rep(A : H))$, $\mathcal{C}^{s^*}(rep(A : H))$ and $\mathcal{C}^w(rep(A : H))$ the respective sets of nondegenerately continuous fields.

Remark 3.14. These notions of continuity cannot apparently be expressed as continuity of certain maps between topological spaces. However, we can describe them as continuity of certain maps at certain points. For example, a field T over $rep(A : H)$ is s^*-nondegenerately continuous if and only if for

every $\pi \in rep(A:H)$ and $\alpha \in \overline{\pi(A)H}$, the maps

$$rep(A:H) \longrightarrow H$$
$$\pi' \longmapsto T(\pi')\alpha$$
$$\pi' \longmapsto T(\pi')^*\alpha$$

are continuous at π. Thus, they are genuine continuity notions.

Recall the following descriptions of left multipliers, right multipliers, multipliers and quasi-multipliers of a C^*-algebra A:
$LM(A) = \{T \in W^*(A) /\ Ta \in A\ \forall a \in A\}$
$RM(A) = \{T \in W^*(A) /\ aT \in A\ \forall a \in A\}$
$M(A) = \{T \in W^*(A) /\ Ta \in A \wedge aT \in A\ \forall a \in A\}$
$QM(A) = \{T \in W^*(A) /\ aTb \in A\ \forall a, b \in A\}$

Applying Takesaki-Bichteler duality it is easy to prove the following theorem.

Theorem 3.15.
1) $\mathcal{C}^s(rep(A:H)) = LM(A)$
2) $\mathcal{C}^{as}(rep(A:H)) = RM(A)$
3) $\mathcal{C}^{s*}(rep(A:H)) = M(A)$
4) $\mathcal{C}^w(rep(A:H)) = QM(A)$

Proof. 1) Take $T \in LM(A)$. We must see that T is s-nondegenerately continuous. If $\pi_j \to \pi$ is a convergent net in $rep(A:H)$, and $w = \pi(a)v \in \pi(A)H$,

$$||(T(\pi_j) - T(\pi))w|| \leq$$
$$\leq ||(T(\pi_j)\pi(a)v - T(\pi_j)\pi_j(a)v|| + ||T(\pi_j)\pi_j(a)v - T(\pi)\pi(a)v|| \leq$$
$$||T||\epsilon' + ||(Ta(\pi_j) - Ta(\pi))v|| \leq \epsilon$$

In the last line we used $Ta \in A$. Elements $\pi(a)v$ generate $\overline{\pi(A)H}$, therefore we can reach $||(T(\pi_j) - T(\pi))w|| \leq \epsilon$ for any $w \in \overline{\pi(A)H}$, i.e. $T \in \mathcal{C}^s(rep(A:H))$.

For the reverse inclusion, the proof is very similar: take $T \in \mathcal{C}^s(rep(A:H))$, and $a \in A$. Let us see that $Ta \in \mathcal{C}_0(rep(A:H))$, which is equal to A by 3.11. Let $\pi_j \to \pi$ be a convergent net of representations.

$$||(Ta(\pi) - Ta(\pi_j))v|| = ||(T(\pi)\pi(a) - T(\pi_j)\pi_j(a))v|| \leq$$

$$\leq ||(T(\pi)\pi(a) - T(\pi_j)\pi(a))v|| + ||T(\pi_j)(\pi(a) - \pi_j(a))v|| \leq$$
$$\epsilon_1 + ||T||\epsilon_2 \leq \epsilon$$

where the first term is small thanks to the continuity hypothesis for T and the second because $\pi_j \to \pi$. This shows $Ta \in A$, or $T \in LM(A)$.

2) follows from $LM(A)^* = RM(A)$, $\mathcal{C}^s(rep(A:H))^* = \mathcal{C}^{as}(rep(A:H))$ and 1).

3) follows from 1), 2) and

$$M(A) = LM(A) \cap RM(A)$$
$$\mathcal{C}^{s^*}(rep(A:H)) = \mathcal{C}^s(rep(A:H)) \cap \mathcal{C}^{as}(rep(A:H))$$

4) The idea from 1) applies. Start with a $T \in QM(A)$ and consider $\pi_j \to \pi$, $\pi(a)v$, $\pi(b)w$. w-nondegenerately continuity of T follows from next computation.

$$|\langle T(\pi_j)\pi(a)v, \pi(b)w\rangle - \langle T(\pi)\pi(a)v, \pi(b)w\rangle| \leq$$
$$|\langle T(\pi_j)\pi(a)v, \pi(b)w\rangle - \langle T(\pi_j)\pi_j(a)v, \pi(b)w\rangle| +$$
$$+|\langle T(\pi_j)\pi_j(a)v, \pi(b)w\rangle - \langle T(\pi_j)\pi_j(a)v, \pi_j(b)w\rangle| +$$
$$+|\langle b^*Ta(\pi_j)v, w\rangle - \langle b^*Ta(\pi)v, w\rangle| \leq$$
$$||T||.\epsilon_1.||b||.||w|| + ||T||.||a||.||v||.\epsilon_2 + \epsilon_3$$

And similarly, if $T \in \mathcal{C}^w(rep(A:H))$, we get $aTb \in A$, for every $a, b \in A$. □

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